Instability of Hollow Beams*

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Summary—This paper gives a linearized theory for the breakup of magnetically focused hollow beams of electrons. A zero-thickness developed beam is assumed. Growing waves are found to be possible both at zero frequency and at finite frequencies. When there are electrodes close to the beam inside and out, the waves grow more rapidly as the electrodes are moved further from the beam. If the current is increased and the magnetic field is increased just enough to keep the beam together, the rate of growth is increased. The theory predicts a greater rate of growth for a higher number n of cycles of variation around the circumference of the beam, but for actual beams of finite thickness the theory is inaccurate for large values of n. In a simple case, frequency becomes important only when the wavelength of waves along the beam becomes comparable with the wavelength measured around the circumference of the beam. Increasing waves are also found in a zero-thickness beam in crossed electric and magnetic fields, as in a magnetron amplifier or carcinotron, but not at zero frequency.

I. The Problem and Some Discussion

THIS PAPER deals with thin hollow electron beams confined by a uniform magnetic field. Interest in the question of the stability of such beams has been aroused by experiments carried out recently by H. F. Webster[1] and later at the Bell Telephone Laboratories by C. C. Cutler and J. T. Mendel. In these experiments a hollow beam, as observed by its trace on a fluorescent screen, is seen to break up into separate beams under certain conditions. As the beam current is increased from a very low value the initially circular trace departs from circular symmetry, both in shape and brightness. Further increase causes the beam to curl up in a number of places, the curls eventually becoming separate beams. The number of beams depends on the current density and the magnetic field.

This phenomenon appears not to depend on the gas pressure, or secondary electrons, or any other trivial causes. Therefore, it appears desirable to examine whether the cause may not be found in a natural instability, not heretofore considered.

In this paper the linear theory of a zero-thickness developed cylindrical beam (a flat sheet of electron flow) is considered. Growing waves are found. In Section III it is shown that when there are electrodes close to the beam inside and outside of it, the rate of growth is greater as the electrodes are moved further away from the beam. In Section IV the case of an unshielded beam is treated; growing waves are found in this case. In Section V the displacement and bunching of the charge are investigated for a particular case. It is found to be in accord with elementary ideas, and to be qualitatively in accord with experiment in so far as can be expected of a linear theory.

The theory discloses not only zero-frequency growing waves, but also growing waves at finite frequencies. In fact, there can be growing waves at finite frequencies when there are no zero-frequency growing waves.

Thus, growing waves of the sort disclosed by the analysis might be useful in amplifiers. They may also explain the growth of noise on electron beams. The

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[1] H. F. Webster, "Breakup of hollow electron beams," *J. Appl. Phys.*, vol. 26, pp. 1386–1387; November, 1955.

possibility of similar growing waves in solid beams has not been adequately investigated.

In Section VI a few remarks are made concerning the behavior of thin beams in crossed electric and magnetic fields, as in the "static" magnetron. Gain is found, but not at zero frequency.

Let us now turn to the particular assumptions underlying the theoretical work in this memorandum. Any waves associated with the finite thickness of the beam (slipping stream effects for instance) will be disregarded. An approximate solution to the problem is obtained by a) assuming a sheet beam of zero thickness, b) by unrolling or developing the beam, so that the beam of radius a becomes a strip beam of width $2\pi a$ in the x direction, to which a periodic boundary condition is applied. Solutions which vary with time and with distance in the x direction and the z direction (the direction of propagation) as

$$e^{j\omega t}e^{-j\beta z}e^{-j\gamma x} \tag{1}$$

are then sought. Because solutions must be periodic around the beam,

$$\gamma = \frac{n}{a} \tag{2}$$

where a is beam radius and n is a positive or negative integer.

II. The Equations of Motion

Let

$$W = \omega - \beta u_0 - \gamma u_1. \tag{3}$$

Here u_0 is the average velocity in the z direction and u_1 is the average velocity in the x direction.

The equations of motion are

$$\ddot{x} = jW\dot{x} = \frac{e}{m}\frac{\partial V}{\partial x} + \omega_c\dot{y} \tag{4}$$

$$\ddot{y} = jW\dot{y} = \frac{e}{m}\frac{\partial V}{\partial y} - \omega_c\dot{x} \tag{5}$$

$$\ddot{z} = jW\dot{z} = \frac{e}{m}\frac{\partial V}{\partial z}. \tag{6}$$

Here e/m, the charge-to-mass ratio of the electron, is taken as a positive quantity, and ω_c is the cyclotron frequency, $(e/m)B$.

The meaning of $\partial V/\partial x$, $\partial V/\partial y$, and $\partial V/\partial z$ at the beam is not entirely obvious because these quantities are discontinuous in crossing the sheet of charge constituting the beam. It is obvious that $\partial V/\partial y$ will be discontinuous if there is an ac component of charge, and $\partial V/\partial x$ and $\partial V/\partial z$ are discontinuous because the charged surface of the beam is wavy.

Let V_1 be the potential above the beam and let V_2 be the potential below the beam. V_1 and V_2 must be equal at the position of the beam, which is displaced an ac amount $\tilde{y}$ from the beam position in the absence of a signal.

Let us now consider the gradients, and for instance, the gradient in the y direction. We can divide $\partial V/\partial y$ into two components; $(\partial V/\partial y)_s$, which is the same on both sides of the beam, and $(\partial V/\partial y)_d$, which is different on the two sides of the beam. The component $(\partial V/\partial y)_d$ must be the part of the gradient due to the local charge, a part which exerts no net force on the local charge. This is especially apparent if we imagine the beam to be made up of separate line charges parallel to the z axis, for we see that such a charge cannot produce a component of field which is the same on both sides of it. Thus, it seems clear that $(\partial V/\partial y)_s$ must be the part of the gradient due to charge remote from the point in question, the part of the gradient which acts on the local charge and which should be used in (4)–(6).

The two components of the gradient are given by the expressions

$$\left(\frac{\partial V}{\partial y}\right)_d = \frac{1}{2}\left(\frac{\partial V_1}{\partial y} - \frac{\partial V_2}{\partial y}\right) \tag{7}$$

$$\left(\frac{\partial V}{\partial y}\right)_s = \frac{1}{2}\left(\frac{\partial V_1}{\partial y} + \frac{\partial V_2}{\partial y}\right). \tag{8}$$

Here $\partial V_1/\partial y$ and $\partial V_2/\partial y$ are evaluated just next to the beam.

We use a similar argument in respect to $\partial V/\partial x$ and $\partial V/\partial z$ in obtaining the component of the gradient which acts on the charge.

We must also consider the equation of continuity, that is, that the divergence of the current density is equal to the negative of the time derivative of the charge density. Let σ be the charge density per unit area and σ_0 be the average value of σ. Then,

$$j\omega\sigma = j\beta(\sigma u_0 + \sigma_0\dot{z}) + j\gamma(\sigma u_1 + \sigma_0\dot{x}) \tag{9}$$

$$\sigma = \frac{\beta\sigma_0}{W}\dot{z} + \frac{\gamma\sigma_0}{W}\dot{x}. \tag{10}$$

Here, σ_0, is a negative quantity.

III. Shields Close to the Beam

Suppose that there are two plane electrodes a distance d above and below the developed beam. The dc field in the y direction will be governed by the voltage V_d applied between the electrodes. If $(\partial V/\partial y)_{s0}$ is

$$\left(\frac{\partial V}{\partial y}\right)_{s0} = \frac{V_d}{2d} \tag{11}$$

and if

$$\dot{y} = 0 \tag{12}$$

then from the dc equation analogous to (5)

$$\frac{e}{m}\frac{V_d}{2d} = \omega_c u_1$$

$$u_1 = \frac{e}{m}\frac{V_d}{2\omega_c d}. \tag{13}$$

Suppose that

$$|\gamma d| \ll 1$$
$$|\beta d| \ll 1. \tag{14}$$

Then, we can regard the beam and electrodes in the light of a parallel plate condenser. The field at a given point will be determined by the displacement and charge at that point only.

If the neutral position of the beam is midway between the two electrodes, the change in voltage at the beam due to a displacement $\tilde{y}$ in the y direction is zero to the first order. However, it is clear that

$$\left(\frac{\partial V_1}{\partial y}\right)(d - \tilde{y}) + \left(\frac{\partial V_2}{\partial y}\right)(d + \tilde{y}) = V_d \tag{15}$$

$$\frac{\partial V_1}{\partial y} - \frac{\partial V_2}{\partial y} = -\frac{(\sigma_0 + \sigma)}{\epsilon}. \tag{16}$$

Whence

$$\frac{\partial V_1}{\partial y} = \frac{V_d}{2d} - \frac{\sigma_0}{2\epsilon} - \frac{\sigma_0 \tilde{y}}{2\epsilon d} - \frac{\sigma}{2\epsilon} - \frac{\sigma \tilde{y}}{2\epsilon d} \tag{17}$$

$$\frac{\partial V_2}{\partial y} = -\frac{V_d}{2d} + \frac{\sigma_0}{2\epsilon} - \frac{\sigma_0 \tilde{y}}{2\epsilon d} + \frac{\sigma}{2\epsilon} - \frac{\sigma \tilde{y}}{2\epsilon d}. \tag{18}$$

To the first order, the ac quantity $(\partial V/\partial y)_s$ is thus

$$\left(\frac{\partial V}{\partial y}\right)_s = -\frac{\sigma_0 \tilde{y}}{2\epsilon d} = -j\frac{\omega_0^2 \dot{y}}{\frac{e}{m} W}. \tag{19}$$

Here

$$\omega_0^2 = \frac{-\frac{e}{m}\sigma_0}{2d\epsilon}. \tag{20}$$

It is easy to see that

$$V = \frac{\sigma d}{2\epsilon}. \tag{21}$$

Thus

$$\left(\frac{\partial V}{\partial x}\right)_s = \frac{-j\gamma\sigma d}{2\epsilon} = \frac{j\beta\gamma d^2\omega_0^2}{\frac{e}{m}W}\dot{z} + \frac{j\gamma^2 d^2\omega_0^2}{\frac{e}{m}W}\dot{x} \tag{22}$$

$$\left(\frac{\partial V}{\partial z}\right)_s = \frac{-j\beta\sigma d}{2\epsilon} = \frac{j\beta^2 d^2\omega_0^2}{\frac{e}{m}W}\dot{z} + \frac{j\beta\gamma d^2\omega_0^2}{\frac{e}{m}W}\dot{x}. \tag{23}$$

We can now combine (19), (22), and (23) with the equations of motion, (4)–(6), and obtain

$$\left(W - \frac{\gamma^2 d^2\omega_0^2}{W}\right)\dot{x} - \frac{\beta\gamma d^2\omega_0^2}{W}\dot{z} + j\omega_c\dot{y} = 0 \tag{24}$$

$$\left(W + \frac{\omega_0^2}{W}\right)\dot{y} = j\omega_c\dot{x} \tag{25}$$

$$\left(W - \frac{\beta^2 d^2\omega_0^2}{W}\right)\dot{z} = \frac{\beta\gamma d^2\omega_0^2}{W}\dot{x}. \tag{26}$$

From (24)–(26) we obtain

$$\frac{W^2 - (\gamma^2 + \beta^2)d^2\omega_0^2}{W^2 - \beta^2 d^2\omega_0^2} = \frac{\omega_c^2}{W^2 + \omega_0^2}. \tag{27}$$

We should perhaps note that if γ is zero

$$W = \pm\sqrt{\omega_c^2 - \omega_0^2}. \tag{28}$$

When ω_0^2 is greater than ω_c^2, if we displace the beam from the central position the image force toward the nearest wall is great enough to overcome the effect of the magnetic field and the beam flows to the wall. This is an unreal condition, of course. Imagine that the beam does not have zero thickness but rather has a thickness b. Then the plasma frequency ω_p is given by

$$\omega_p^2 = \frac{-\frac{e}{m}\sigma_0}{b\epsilon} = 2\frac{d}{b}\omega_0^2. \tag{29}$$

However, we know that

$$\omega_p^2 \leq \omega_c^2. \tag{30}$$

(In two-dimensional Brillouin flow the equality holds.) Hence,

$$\omega_0^2 \leq \frac{b}{2d}\omega_c^2. \tag{31}$$

Let us now consider (27) in the case, in which $\omega_0 d$ is small, so that

$$\omega_0 d < u_0$$
$$\frac{\omega_0}{u_0}d < 1. \tag{32}$$

In this case, for large values of β, W^2 will increase more rapidly than $(\beta\, d\omega_0)^2$, and as we vary β, the plot of the left-hand side of (27) vs W will look as shown by the solid curves of Fig. 1. The dashed curves show the right-hand

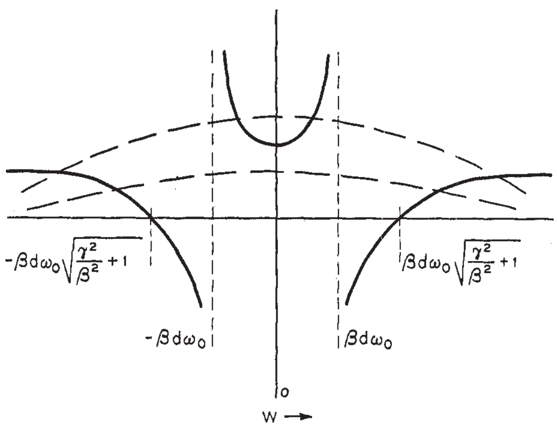

Fig. 1—The two sides of the characteristic equation, indicating conditions for increasing waves, when the dashed curve fails to intercept the solid curve at the center, increasing waves are present.

side of (27) vs W for two values of ω_c. We see that the dashed curve will fail to cut the solid curve near $W = 0$ if

$$1 + \frac{\gamma^2}{\beta^2} > \frac{\omega_c^2}{\omega_0^2}$$

$$\omega_0^2 > \frac{\omega_c^2}{1 + \gamma^2/\beta^2}. \tag{33}$$

This is the condition for complex roots and increasing waves.

If we put in (33) the value of β which makes W in (3) equal to zero, and if we express γ by means of (2), we have as the condition for increasing waves approximately

$$\omega_0^2 > \frac{\omega_c^2\left(\frac{\omega}{u_0} - \frac{n}{a}\frac{u_1}{u_0}\right)^2}{\left(\frac{n}{a}\right)^2}. \tag{34}$$

We see that there will always be increasing waves near

$$\frac{\omega a}{u_0} = n\frac{u_1}{u_0}.$$

For zero frequency we have[2]

$$\omega_0^2 > \left(\frac{\omega_c u_1}{u_0}\right)^2. \tag{35}$$

In this case it appears that a small value of u_1 favors the appearance of growing waves, and there is always a growing wave if u_1 is zero.

Let us now assume that (33) is satisfied, so that there are growing waves. Let

$$W = jP. \tag{36}$$

Then, (27) becomes

$$\frac{P^2 + (\gamma^2 + \beta^2)d^2\omega_0^2}{P^2 + \beta^2 d^2\omega_0^2} = \frac{\omega_c^2}{\omega_0^2 - P^2}. \tag{37}$$

In Fig. 2, the right-hand side of (37) is plotted as a solid line and the left-hand side as dashed line. Curve 2 is for a smaller value of $d^2\omega_0^2$ than is curve 1. We see that as $d^2\omega_0^2$ is made smaller the magnitude of P at the intersection is made smaller. This means that as $d^2\omega_0^2$ is decreased the growing wave will not grow so fast. We should note that

$$d^2\omega_0^2 = \left(\frac{-\frac{e}{m}\sigma_0}{2\epsilon}\right)d. \tag{38}$$

Thus, for a given charge density (a given current and voltage) increasing d tends to make the wave grow faster.

The expressions we have used for the fields [(19), (22), (23)] hold only when γd is small. Hence, if we want to explore the case of remote electrodes or, in the limit, that of an unshielded beam, we must use more realistic expressions for the fields.

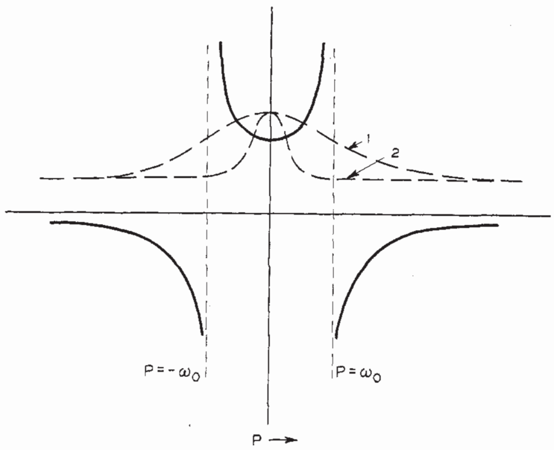

Fig. 2—The characteristic equation for a beam with no electrodes near it plotted in a different manner from that of Fig. 1. Here intersections indicate increasing waves. Intersections further to the left or right mean a greater rate of increase or decrease. Curve 1 is for a larger current than for curve 2.

IV. The Case of No Shield

Let us now assume that together with the already prescribed variations with respect to x and z, above the beam the potential is

$$V_1 e^{-ky} - \alpha\frac{\sigma_0}{\epsilon}y \tag{39}$$

and below the beam the potential is

$$V_2 e^{ky} + (1 - \alpha)\frac{\sigma_0}{\epsilon}y. \tag{40}$$

If Laplace's equation is to be satisfied we must have

$$k^2 = \beta^2 + \gamma^2. \tag{41}$$

If the potentials above and below the beam are to be equal to the first order, we must have

$$V_1 - \alpha\frac{\sigma_0}{\epsilon}\tilde{y} = V_2 + (1 - \alpha)\frac{\sigma_0}{\epsilon}\tilde{y}$$

$$V_1 - V_2 = \frac{\sigma_0}{\epsilon}\tilde{y}. \tag{42}$$

We also note that the gradient in the y direction which acts on the charge is

$$\left(\frac{\partial V}{\partial y}\right)_s = -\frac{k}{2}(V_1 - V_2) = -\frac{k\sigma_0}{\epsilon}\tilde{y} \tag{43}$$

$$\left(\frac{\partial V}{\partial y}\right)_s = -\frac{jka\omega_1^2}{\frac{e}{m}W}\dot{y}. \tag{44}$$

Here

$$\omega_1^2 = \frac{-\frac{e}{m}\sigma_0}{2a\epsilon}. \tag{45}$$

[2] R. L. Kyhl has pointed out to the author that (35) and a later condition, (56) represent the following physical situation: when waves travel at an appreciable angle with respect to the axis of the beam and of the magnetic field, a debunching force exists in the z direction. There is no mechanism present to counteract this debunching, and if it becomes large enough the gain is suppressed.

To the first order, we have from Gauss's theorem

$$-k(V_1 + V_2) = \frac{-\sigma}{\epsilon}. \qquad (46)$$

From (46) and (10) we obtain

$$V_1 + V_2 = -\frac{2\beta a\omega_1^2}{(e/m)kW}\dot{z} - \frac{2\gamma a\omega_1^2}{(e/m)kW}\dot{x}. \qquad (47)$$

Thus, we will obtain

$$\left(\frac{\partial V}{\partial z}\right)_s = \frac{j\beta^2 a\omega_1^2}{(e/m)kW}\dot{z} + \frac{j\beta\gamma a\omega_1^2}{(e/m)kW}\dot{x} \qquad (48)$$

$$\left(\frac{\partial V}{\partial x}\right)_s = \frac{j\beta\gamma a\omega_1^2}{(e/m)kW}\dot{z} + \frac{j\gamma^2 a\omega_1^2}{(e/m)kW}\dot{x}. \qquad (49)$$

We see that (44), (48), and (49) are the same in form as (19), (23), and (22), and if we substitute $ka\omega_1^2$ where ω_0^2 occurs in (27) and $\omega_1^2 a/k$ wherever $\omega_0^2\, d^2$ occurs in (27) we obtain the correct characteristic equation for the new case. This gives

$$\frac{W^2 - (\gamma^2 + \beta^2)\omega_1^2 a/k}{W^2 - \omega_1^2\beta^2 a/k} = \frac{\omega_c^2}{W^2 + ka\omega_1^2}. \qquad (50)$$

Using (41), we obtain

$$\frac{W^2 - ka\omega_1^2}{W^2 - (\beta^2 a/k)\omega_1^2} = \frac{\omega_c^2}{W^2 + ka\omega_1^2}$$

$$W^4 - \omega_c^2 W^2 - \omega_c^4\left[\left(\frac{ka\omega_1^2}{\omega_c^2}\right)^2 - \frac{\beta^2 a\omega_1^2}{k\omega_c^2}\right] = 0. \qquad (51)$$

We can solve this for W and write

$$W = \pm\omega_c\left[\frac{1 \pm \sqrt{1 + 4\left(\frac{ka\omega_1^2}{\omega_c^2}\right)^2\left(1 - \frac{\beta^2\omega_c^2}{k^3 a\omega_1^2}\right)}}{2}\right]^{1/2}. \qquad (52)$$

If we regard β as having so small an imaginary component that we can treat it as real in the following expressions, for the negative sign in the brackets there will be complex roots if

$$\frac{k^3 a\omega_1^2}{\beta^2\omega_c^2} = \frac{1}{4}\frac{k^2}{\beta^2}\left(4\frac{ka\omega_1^2}{\omega_c^2}\right) > 1. \qquad (53)$$

We further find that in physically reasonable cases

$$4\left(\frac{ka\omega_1^2}{\omega_c^2}\right) \ll 1. \qquad (54)$$

If (53) and (54) are to be so, then we must have

$$\frac{1}{4}\frac{k^2}{\beta^2} \gg 1. \qquad (55)$$

Thus, from (55), (41), and (3) we must have very nearly

$$\frac{1}{4}\left(1 + \left(\frac{u_0}{u_1}\right)^2\right) \gg 1. \qquad (56)$$

Let us see what (54) implies. We have noted in (30) that for two-dimensional Brillouin flow the plasma frequency is equal to the cyclotron frequency, and for other flows it is less than the cyclotron frequency. Hence we have for an actual beam which has some thickness b

$$\omega_p^2 = \frac{2a}{b}\omega_1^2 \leq \omega_c^2$$

$$\frac{\omega_1^2}{\omega_c^2} \leq \frac{b}{2a}. \qquad (57)$$

We should also note that as from (55) $k^2 \gg \beta^2$, then from (41) and (2), very nearly

$$k^2 = \gamma^2 = \left(\frac{n}{a}\right)^2. \qquad (58)$$

Thus, according to (57) and (58), (54) will be true if

$$\frac{2nb}{a} \ll 1. \qquad (59)$$

Clearly, our analysis, which is based on a beam of zero thickness, will be inapplicable unless b is very small compared with a, and hence it can be used only if (59), and hence (54), are satisfied.

Assuming the negative sign in (52), which leads to increasing waves, assuming (54), and using (56), we find that approximately

$$W = \omega - \beta u_0 - \gamma u_1 = \pm j\frac{n\omega_1^2}{\omega_c}\left(1 - \frac{\beta^2 a^2\omega_c^2}{n^3\omega_1^2}\right)^{1/2}$$

$$\beta = \frac{\omega}{u_0} - \gamma\frac{u_1}{u_0} \pm j\frac{n\omega_1^2}{\omega_c u_0}\left(1 - \frac{\beta^2 a^2\omega_c^2}{n^3\omega_1^2}\right) \qquad (60)$$

When W is small, very nearly

$$\beta = \frac{\omega}{u_0}. \qquad (61)$$

If we use this value of β in the radical in (60) we obtain

$$\beta = \frac{\omega}{u_0} - \frac{nu_1}{au_0} \pm j\frac{n\omega_1^2}{\omega_c u_0}\sqrt{1 - \frac{\omega^2 a^2\omega_c^2}{n^3 u_0^2\omega_1^2}}. \qquad (62)$$

We note that the gain will go to zero at a frequency such that

$$\left(\frac{\omega}{\omega_c}\right)^2 = n^3\left(\frac{u_0}{\omega_1 a}\right)^2. \qquad (63)$$

It is of some interest to express ω_1 in terms of beam current and ω_c in terms of magnetic field. The average beam current I_0 is

$$I_0 = 2\pi a\sigma_0 u_0. \qquad (64)$$

In terms of beam voltage V_0

$$u_0^2 = 2\frac{e}{m}V_0. \qquad (65)$$

From (45), (63), (65), and (60) we have the zero frequency

$$|Im\,\beta| = \frac{nI_0}{8\pi\epsilon(e/m)V_0 Ba^2}. \qquad (66)$$

In a length L the gain G in db will be

$$G = \frac{20}{\ln 10}|Im\,\beta|L\,db. \qquad (67)$$

Numerically

$$G = .225 \frac{nI_0L}{V_0Ba^2}. \tag{68}$$

Further, we can express (63), giving the frequency at which the gain goes to zero, as

$$\left(\frac{\omega}{\omega_c}\right)^2 = 8\sqrt{2}\,\pi\epsilon(e/m)^{1/2}\frac{n^3V_0^{3/2}}{I_0}. \tag{69}$$

Numerically, this is

$$\left(\frac{\omega}{\omega_c}\right)^2 = 1.32 \times 10^{-4}\frac{nV_0^{3/2}}{I_0}. \tag{70}$$

V. The Motions of the Electrons

Let us now consider in a particular case how the electrons move and bunch as the wave grows.

We have from (25) and (26)

$$\dot{x} = \frac{-j(W^2 + \omega_0^2)}{\omega_c W}\,\dot{y} \tag{71}$$

$$\tilde{x} = \frac{-j(W^2 + \omega_0^2)}{\omega_c W}\,\tilde{y} \tag{72}$$

and

$$\dot{z} = -j\,\frac{\beta\gamma d^2\omega_0^2}{\omega_c W}\,\frac{(W^2 - \omega_0^2)}{(W^2 - \beta^2 d^2\omega_0^2)}. \tag{73}$$

In going to the case of the tubular beam without electrodes, as before we replace ω_0^2 by $ka\omega_1^2$ and $\omega_0^2\, d^2$ by $\omega_1^2 a/k$. This leads to

$$\dot{x} = \frac{-j(W^2 + ka\omega_1^2)}{\omega_c W}\,\dot{y} \tag{74}$$

$$\tilde{x} = \frac{-j(W^2 + ka\omega_1^2)}{\omega_c W}\,\tilde{y} \tag{75}$$

$$\dot{z} = \frac{-j\beta\gamma\omega_1^2 a(W^2 + ka\omega_1^2)}{k\omega_c W\left(W^2 - \dfrac{\beta^2\omega_1^2 a}{k}\right)}\,\dot{y}. \tag{76}$$

We have in addition

$$\sigma = \frac{\sigma_0}{W}(\beta\dot{z} + \gamma\dot{x}). \tag{77}$$

Let us now consider W as given by (60). Let us assume, as will be true at low frequencies, that

$$\frac{\beta^2 a^2\omega_c^2}{n^3\omega_1^2} \ll 1. \tag{78}$$

This means that we can neglect this quantity in (52). Thus, we have for the increasing wave

$$W = \frac{-jn\omega_1^2}{\omega_c} \tag{79}$$

$$W^2 = -\frac{n^2\omega_1^4}{\omega_c^2}. \tag{80}$$

Accordingly

$$\dot{x} = \left(1 - \frac{n\omega_1^2}{\omega_c^2}\right)\dot{y} \tag{81}$$

$$\tilde{x} = \left(1 - \frac{n\omega_1^2}{\omega_c^2}\right)\tilde{y} \tag{82}$$

$$\dot{z} = -\frac{\beta\gamma}{k^2}\,\frac{\left(1 - \dfrac{n\omega_1^2}{\omega_c^2}\right)}{\left(\dfrac{n\omega_1^2}{\omega_c^2} + \dfrac{\beta^2 a^2}{n^2}\right)}\,\dot{y}. \tag{83}$$

Because of (78), the first term in the denominator of (82) is much larger than the second, and very nearly

$$\dot{z} = -\left(\frac{\beta^2 a^2\omega_c^2}{n^3\omega_1^2}\right)\left(\frac{\gamma}{\beta}\right)\left(1 - \frac{n\omega_1^2}{\omega_c^2}\right)\dot{y}. \tag{84}$$

From (10)

$$\sigma = \frac{j\gamma\sigma_0\gamma\omega_c}{n\omega_1^2}\left(1 - \frac{n\omega_1^2}{\omega_c^2}\right)\left(1 - \frac{\beta^2 a^2\omega_c^2}{n^3\omega_1^2}\right)\dot{y}. \tag{85}$$

From (78) we see that we can disregard the term obtained from $\dot{z}$ in comparison with that arising from $\dot{x}$, and write

$$\sigma = \frac{j\sigma_0\gamma\omega_c}{n\omega_1^2}\left(1 - \frac{n\omega_1^2}{\omega_c^2}\right)\dot{y}. \tag{86}$$

The bunching of electrons associated with $\dot{z}$ is negligible compared with that associated with $\dot{x}$.

We also have from (54) and (58)

$$\frac{n\omega_1^2}{\omega_c^2} \ll 1. \tag{87}$$

When this is so, we can write

$$\dot{x} = \dot{y} \tag{88}$$

$$\tilde{x} = \tilde{y}. \tag{89}$$

By using (86), (79), and (87), we can rewrite (86)

$$\sigma = j\sigma_0\frac{n}{a}\,y. \tag{90}$$

Thus, if y as a function of x is the real part of

$$Ae^{-j(n/a)x}. \tag{91}$$

Then

$$\tilde{y} = A\cos\left(\frac{n}{a}x\right) \tag{92}$$

$$\tilde{x} = A\cos\left(\frac{n}{a}x\right) \tag{93}$$

$$\sigma = \sigma_0\frac{n}{a}A\sin\left(\frac{n}{a}x\right). \tag{94}$$

In Fig. 3, x and y have been shown graphically. Displaced particles of charge are shown as dots, connected by slanting (45°) lines with the undisplaced positions of the particles on the y axis. We will note that the charges are bunched together to the right of $x = 0$ and spread apart to the left of $x = 0$, in accord with (94). The mechanism is clear. Consider (4) and (5) for the case of a radial outward gradient and a cylinder of charge smoothly rotating, so that $\beta = \gamma = \omega = W = 0$. We see that the rotation will be clockwise. Now, in Fig. 3, because of the bunching of negative charge about $nx/a = \pi/2$, there is a positive gradient away from this point (or line). Hence, this tends to make the charge on which it acts rotate clockwise about the point, as indicated. When the motion is considered in more detail (as it was in Section IV, from which the displacements of Fig. 3 were deduced) the motion is found to be such as to produce bunching around $nx/a = (n + 1/2)\pi$.

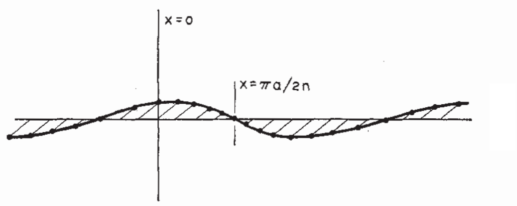

Fig. 3—The displacement of electrons around the circumference of the beam for a growing wave. The motions are those to be expected due to the forces produced by the bunching and displacement of the charge.

Thus, we have shown in a particular and simple case that the electron motions deduced are in accord with elementary ideas.

The calculations made in this memorandum are based on a linearized theory. Actual observation shows that eventually the electrons wind up in a pattern somewhat as shown in Fig. 4, and even more so.

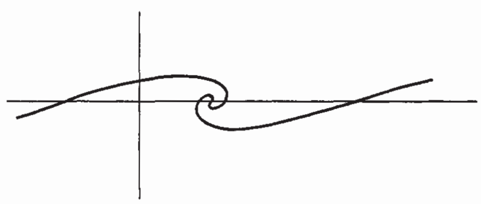

Fig. 4—The presumed nonlinear development of the process indicated in Fig. 3, which corresponds to experimentally observed effects.

VI. Linear Zero-Thickness Beam

We should note that the flow of a sheet beam in crossed electric and magnetic fields, as in a "static" magnetron, can be investigated by letting

$$\beta = u_0 = 0. \tag{95}$$

In this case, the phase constant for propagation in the direction of electron motion is γ, and further

$$k = \gamma \tag{96}$$

$$W = \omega - \gamma u_1. \tag{97}$$

Eq. (51), which holds for remote electrodes (in the sense of $1/\gamma$ as a critical dimension) reduces to

$$W^4 - \omega_c^2 W^2 - \gamma^2 a^2 \omega_1^4 = 0 \tag{98}$$

and corresponding to (60) we have

$$\gamma = \frac{\omega}{u_1} \pm j\,\frac{\gamma a \omega_1^2}{\omega_c u_1}. \tag{99}$$

In the right-hand side of (99) we approximate γ by ω/u_1 and obtain

$$\gamma = \frac{\omega}{u_1}\left(1 + j\left(\frac{\omega_1}{\omega_c}\right)\frac{\omega_1 a}{u_1}\right) \tag{100}$$

These growing waves must be the diochotron effect.[3]

If in (98) we consider the case

$$\omega = 0 \tag{101}$$

we obtain:

$$\gamma^4 - \gamma^2\left(\left(\frac{\omega_c}{u_1}\right)^2 + \left(\frac{\omega_1}{u_1}\right)^4 a^2\right) = 0. \tag{102}$$

This leads to no increasing waves.

If we return to (27) we find for zero frequency

$$\gamma = \pm\frac{1}{u_1}\left(\frac{\omega_c^2}{1 - (\omega_0 d/u_1)^2} - \omega_0^2\right)^{1/2}. \tag{103}$$

This can lead to growing waves only if

$$\left(\frac{\omega_0 d}{u_1}\right)^2 > 1. \tag{104}$$

From (22) this requires that

$$\frac{1}{2}\left(\frac{\omega_c b}{2u_1}\right)\left(\frac{\omega_c d}{2u_1}\right) > 1. \tag{105}$$

But, unless $(\omega_c b/2u_1)$ and $(\omega_c d/2u_1)$ are both small compared to 1, our assumptions concerning the thinness of the beam and the nature of the field are violated and hence the equations we have used in obtaining (27) are invalid if (105) is satisfied.

The author is indebted to R. Kompfner and C. C. Cutler for valuable suggestions and comments.

[3] H. Huber and P. Guenard, "Étude expérimentale de l'interaction par ondes de charge d'espace au sein d'un fraisceau électronique se deplacant dans des champs electrique et magnetique croises," *Ann. Radioelect.*, vol. 7, pp. 252–278; October, 1952.